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The monopole-hydrogen atom system and its connection with a four-dimensional harmonic oscillator

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Abstract. The Hamiltonian of a hydrogen atom associated with a $U(1)$ monopole is established in parallel to a usual hydrogen atom with conserved angular momentum **L** and Pauli–Runge–Lenz vector **R**. Both are associated with monopoles and form an algebra $SO(4)$ for the bound motions. Due to the two corresponding Pauli relations and their bosonic realizations, the system can be connected to a four-dimensional harmonic oscillator with a monopole-dependent constraint. Furthermore, the monopole harmonics can be obtained by the operator method.

1. Introduction

The subject of magnetic monopoles began with Dirac's paper on quantized singularities in the electromagnetic field [1]. The motivation for that paper was to find the reason for the existence of the smallest electric charge. After 1931, and before 1948 when Dirac wrote on the subject again, the theory of monopoles and of bound states of monopoles and electric charges was worked on extensively by Tamm, Fierz, and others [2, 3]. Nearly 30 years ago, McIntosh and Cisneros [4], and Zwanziger [5] suggested considering a spinless system (the MIC–Zwanziger system) in a combined monopole plus scalar potential field, described by the Hamiltonian

$$H_0 = \frac{\vec{\pi}^2}{2\mu} + \frac{q^2}{2}\left(1 - \frac{1}{r}\right)^2. \tag{1}$$

Since then the Hamiltonian systems in quantum mechanics associated with monopoles have aroused a great deal of interest [6–9]. For example, D'Hoker and Vinet constructed a Hamiltonian (the D'Hoker–Vinet system)

$$H_1 = H_0 - q\vec{\sigma} \cdot \frac{\mathbf{r}}{r^3} \tag{2}$$

(with $\vec{\sigma}$ the Pauli vector), which describes the dynamics of a charged spin-$\frac{1}{2}$ particle with anomalous gyromagnetic ratio 4 in the field of a dyon. When without a monopole, i.e. $q = 0$, equations (1) and (2) reduce to the Hamiltonian of a free particle.

As is well known, there are two conserved vectors in a hydrogen atom, i.e. the angular momentum $\vec{\ell} = \vec{r} \times \vec{p}$ and the Pauli–Runge–Lenz (PRL) vector $\vec{\mathbf{b}}$, both of them satisfying $\vec{\ell} \cdot \vec{\mathbf{b}} = 0$ and $[h, \vec{\ell}] = [h, \vec{\mathbf{b}}] = 0$, where $h = \frac{p^2}{2\mu} - \frac{\kappa}{r}$ is the Hamiltonian of a hydrogen atom.

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If there is a monopole, it is natural to extend $\vec{\ell}$ and $\vec{\mathbf{b}}$ to become monopole-dependent $\mathbf{L}$ and $\mathbf{R}$ (see equations (3) and (5)), and h to monopole-dependent Hamiltonian H (see equation (11)) commuting with $\mathbf{L}$ and $\mathbf{R}$. However, in this case $\mathbf{L}\cdot\mathbf{R} \neq 0$, later one can see that $\mathbf{L}\cdot\mathbf{R}$ is related to the monopole charge. On the other hand, the problem-solving hydrogen atom ($h\phi = E\phi$) can be transformed into solving a four-dimensional harmonic oscillator with constraint $\vec{\ell}\cdot\vec{\mathbf{b}} = 0$, which may be traced back to the early days of quantum mechanics [10–12] and later discussed by many authors in [13–23]. Now a question is put forward: can we transform the monopole-dependent three-dimensional Coulomb problem to a four-dimensional harmonic oscillator with the extended constraint $\mathbf{L}\cdot\mathbf{R} \neq 0$? To our knowledge, this problem has not been discussed in the literature. The purpose of this paper is to establish the monopole-hydrogen atom system and connect it to a four-dimensional harmonic oscillator with constraint $\mathbf{L}\cdot\mathbf{R} = q\mu\kappa$. In addition, the shift operators for a spherical harmonic function [23] are extended to include monopoles in this paper. The extended ones then shift the Wu–Yang monopole harmonics [24].

This paper is organized as follows. In section 2, we construct the Hamiltonian of the monopole-hydrogen atom H (see equation (11)) from the point of view of $SO(4)$ dynamical symmetry for bound motions. The two corresponding Pauli relations are found and the energy spectrum is also given. In section 3, the generators of the $SO(4)$ algebra are realized by bosonic operators, after substituting them into the Pauli relations, the Hamiltonian system H is then connected to a four-dimensional harmonic oscillator with a monopole-dependent constraint. In the last section, as a supplement, the monopole harmonics are obtained by the operator method.

2. Conserved vectors and energy spectrum

The orbital angular momentum operator associated with the $U(1)$ monopole is introduced by [24]

$$\mathbf{L} = \mathbf{r}\times\vec{\pi} - q\frac{\mathbf{r}}{r} \qquad \vec{\pi} = \mathbf{p} - Ze\mathbf{A} \tag{3}$$

where $\mathbf{A}$ is the vector-potential of a Dirac monopole with strength g, $\nabla\times\mathbf{A} = g\frac{\mathbf{r}}{r^3}$ for either $\mathbf{A}_a$ or $\mathbf{A}_b$ in the region a and b as defined in [24] and

$$q = Zeg = \tfrac{1}{2}\times\text{integer}. \tag{4}$$

It is easy to observe from (3) that $\mathbf{L}$ reduces to the usual orbital angular momentum if $q = 0$. Motivated by the form of the PRL vector for the usual hydrogen atom, we introduce the extended one

$$\mathbf{R} = \tfrac{1}{2}(\vec{\pi}\times\mathbf{L} - \mathbf{L}\times\vec{\pi}) - f(r)\mathbf{r} \tag{5}$$

where $f(r)$ is a real function of r, and is to be determined later by requiring that $\mathbf{L}$ and $\mathbf{R}$ generate a dynamical group $SO(4)$.

Noting that (in unit $\hbar = 1$)

$$\begin{aligned} &[L_\alpha, L_\beta] = \mathrm{i}\epsilon_{\alpha\beta\gamma}L_\gamma \qquad &&[L_\alpha, \pi_\beta] = \mathrm{i}\epsilon_{\alpha\beta\gamma}\pi_\gamma \qquad &&\alpha,\beta,\gamma = x,y,z \\ &[L_\alpha, r_\beta] = \mathrm{i}\epsilon_{\alpha\beta\gamma}r_\gamma \qquad &&[\pi_\alpha, \pi_\beta] = \mathrm{i}\epsilon_{\alpha\beta\gamma}q\frac{r_\gamma}{r^3} \qquad &&[\pi_\alpha, r_\beta] = -\mathrm{i}\delta_{\alpha\beta} \end{aligned} \tag{6}$$

we obtain

$$\mathbf{R}\times\mathbf{R} = -\mathrm{i}\left[\vec{\pi}^2 + \frac{q^2}{r^2} - 3f(r) - rf'(r)\right]\mathbf{L} + \mathrm{i}q\left[f'(r) + \frac{f(r)}{r}\right]\mathbf{r} \tag{7}$$

where $f'(r) = \frac{\mathrm{d}}{\mathrm{d}r}f(r)$.

Let

$$f'(r) + \frac{f(r)}{r} = 0 \tag{8}$$

so that

$$f(r) = \frac{c}{r} \qquad 3f(r) + rf'(r) = \frac{2c}{r} \tag{9}$$

here c is an arbitrary constant. Therefore, (7) becomes

$$\mathbf{R} \times \mathbf{R} = -2\mathrm{i}\mu H\mathbf{L} \tag{10}$$

with

$$H = \frac{\vec{\pi}^2}{2\mu} + \frac{1}{2\mu}\frac{q^2}{r^2} - \frac{\kappa}{r} \tag{11}$$

where μ is the reduced mass of the hydrogen atom and $\kappa = Ze^2$. (Here we have chosen $c = \mu\kappa$ such that H reduces to the Hamiltonian of the usual hydrogen atom when $q = 0$. By the way, if one selects $c = q^2$ instead, from equations (7)–(10) the Hamiltonian H_0 shown in (1) can be derived.)

The direct calculation shows

$$[H, \mathbf{L}] = 0 \qquad [H, \mathbf{R}] = 0. \tag{12}$$

It indicates that in the monopole-hydrogen atom system shown by (11), there also exist two conserved vectors, just like the usual Coulomb problem. The role played by $\mathbf{R}$ is the same as the usual PRL vector $\vec{\mathbf{b}}$ in a hydrogen atom, hence we call $\mathbf{R}$ the monopole-PRL vector.

For the bound states of H, after replacing it by $E(E < 0)$, we have

$$\mathbf{L}\cdot\mathbf{R} = \mathbf{R}\cdot\mathbf{L} = q\mu\kappa \tag{13}$$

and

$$\mathbf{R}^2 - (\mu\kappa)^2 = 2\mu H(\mathbf{L}^2 - q^2 + 1). \tag{14}$$

By rescaling

$$\mathbf{B} = \sqrt{\frac{-1}{2\mu E}}\mathbf{R} \tag{15}$$

equations (13) and (14) recast to

$$\mathbf{L}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{L} = q\sqrt{-\frac{\mu\kappa^2}{2E}} \tag{16}$$

and

$$\mathbf{L}^2 + \mathbf{B}^2 + 1 = q^2 - \frac{\mu\kappa^2}{2E}. \tag{17}$$

Clearly, equations (16) and (17) are the two corresponding Pauli relations for the Hamiltonian H, since when $q = 0$, they will be reduced to the usual ones [25], i.e. (16) and (17) are degenerated.

From equations (6), (10) and (15), it follows the $SO(4)$ algebra spanned by $\mathbf{L}(L_x, L_y, L_z)$ and $\mathbf{B}(B_x, B_y, B_z)$, i.e.

$$[L_\alpha, L_\beta] = \mathrm{i}\epsilon_{\alpha\beta\gamma}L_\gamma \qquad [L_\alpha, B_\beta] = \mathrm{i}\epsilon_{\alpha\beta\gamma}B_\gamma \qquad [B_\alpha, B_\beta] = \mathrm{i}\epsilon_{\alpha\beta\gamma}L_\gamma. \tag{18}$$

Now introducing

$$\mathbf{N} = \tfrac{1}{2}(\mathbf{L} + \mathbf{B}) \qquad \mathbf{M} = \tfrac{1}{2}(\mathbf{L} - \mathbf{B}) \tag{19}$$

one gets

$$\mathbf{N}^2 = \mathbf{M}^2 + q\sqrt{-\frac{\mu\kappa^2}{2E}} = \frac{1}{4}\left[\left(q + \sqrt{-\frac{\mu\kappa^2}{2E}}\right)^2 - 1\right] \tag{20}$$

and

$$[N_\alpha, M_\beta] = 0 \qquad [N_\alpha, N_\beta] = \mathrm{i}\epsilon_{\alpha\beta\gamma}N_\gamma \qquad [M_\alpha, M_\beta] = \mathrm{i}\epsilon_{\alpha\beta\gamma}M_\gamma. \tag{21}$$

It means that $\mathbf{N}$, $\mathbf{M}$ are two independent angular momentums. Let $n(n+1)$ and $m(m+1)$ denote eigenvalues of $\mathbf{N}^2$ and $\mathbf{M}^2$, respectively. On the basis of (20) the energy spectrum is given by

$$E_n = -\frac{\mu\kappa^2}{2}\frac{1}{(2n+1-q)^2} \qquad n = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots \tag{22}$$

or

$$E_m = -\frac{\mu\kappa^2}{2}\frac{1}{(2m+1+q)^2} \qquad m = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots. \tag{23}$$

Obviously if without a monopole, $E_n (= E_m)$ will reduce to the energy of the usual hydrogen atom.

3. Connection with a four-dimensional harmonic oscillator

Without a monopole, extensive discussions about the transformation from the three-dimensional Coulomb problem to a four-dimensional harmonic oscillator with a constraint have been made. The typical example is the Kustaanheimo–Stiefel (KS) transformation [13–23]. In fact, the constraint condition can be shown to be equivalent to the fact that the angular momentum vector and the PRL vector are orthogonal to each other [19–23]. Following the spirit of [19–22], we shall establish the connection between the monopole-hydrogen atom system and a four-dimensional harmonic oscillator directly by introducing the bosonic realizations of $\mathbf{L}$ and $\mathbf{B}$ and using two basic Pauli relations (16) and (17) satisfied by them, i.e. without making use of the KS transformation.

To specify the vectors $\mathbf{L}$ and $\mathbf{B}$ given by (15), we take the following bilinear bosonic realizations

$$L_j = \tfrac{1}{2}(a^+\sigma_j a + b^+\sigma_j b) \qquad B_j = \tfrac{1}{2}(a^+\sigma_j\sigma_y\tilde{b}^+ - \tilde{a}\sigma_y\sigma_j b) \qquad j = x, y, z \tag{24}$$

with

$$a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \qquad b = \begin{pmatrix} a_3 \\ a_4 \end{pmatrix} \qquad a^+ = (a_1^+ \; a_2^+) \qquad \tilde{a} = (a_1 \; a_2) \quad \text{etc} \tag{25}$$

where $\sigma_j (j = x, y, z)$ are Pauli matrices, a_j and a_j^+ $(j = 1, 2, 3, 4)$ are annihilation and creation bosonic operators. By substituting (24) into equations (16) and (17), equation (16) yields

$$(a_1^+a_1 + a_2^+a_2 - a_3^+a_3 - a_4^+a_4)(a_1^+a_4^+ + a_1a_4 - a_2^+a_3^+ - a_2a_3) = \mathrm{i}4q\sqrt{-\frac{\mu\kappa^2}{2E}} \tag{26}$$

and (17) leads to

$$(a_1^+a_1 + a_2^+a_2 - a_3^+a_3 - a_4^+a_4)^2 - (a_1^+a_4^+ + a_1a_4 - a_2^+a_3^+ - a_2a_3)^2 = 4\left(q^2 - \frac{\mu\kappa^2}{2E}\right). \tag{27}$$

(Note that equations (26) and (27), as well as some similar relations in this paper are understood to be acted on by a wavefunction ψ of the system.) On account of the fact that the two factors in the left-hand side of (26) commute to each other, then (26) and (27) give

$$a_1^+a_1 + a_2^+a_2 - a_3^+a_3 - a_4^+a_4 = \pm 2q \tag{28}$$

and

$$a_1^+a_4^+ + a_1a_4 - a_2^+a_3^+ - a_2a_3 = \mp \mathrm{i}2\sqrt{-\frac{\mu\kappa^2}{2E}}. \tag{29}$$

With the help of the standard combinations

$$Q_j = \sqrt{\frac{1}{2\mu\omega}}(a_j + a_j^+) \qquad P_j = -\mathrm{i}\sqrt{\frac{\mu\omega}{2}}(a_j - a_j^+) \tag{30}$$

where $\omega = \sqrt{-E/2\mu} > 0$, (29) can be recast to

$$\frac{1}{2\mu}P_1P_4 - \frac{1}{2}Q_1Q_4 - \frac{1}{2\mu}P_2P_3 + \frac{1}{2}Q_2Q_3 = \pm \mathrm{i}\omega\sqrt{-\frac{\mu\kappa^2}{2E}}. \tag{31}$$

Motivated by [23], the following transformation $\{P_j, Q_j : j = 1, 2, 3, 4\} \to \{P_j', Q_j' : j = 1, 2, 3, 4\}$

$$\begin{aligned}
P_1 &= \frac{1}{\sqrt{2}}(P_1' + \mathrm{i}P_4') \qquad & P_4 &= \frac{1}{\sqrt{2}}(P_1' - \mathrm{i}P_4') \\
Q_1 &= \frac{1}{\sqrt{2}}(Q_1' - \mathrm{i}Q_4') \qquad & Q_4 &= \frac{1}{\sqrt{2}}(Q_1' + \mathrm{i}Q_4') \\
P_2 &= \frac{\mathrm{i}}{\sqrt{2}}(P_2' + \mathrm{i}P_3') \qquad & P_3 &= \frac{\mathrm{i}}{\sqrt{2}}(P_2' - \mathrm{i}P_3') \\
Q_2 &= \frac{-\mathrm{i}}{\sqrt{2}}(Q_2' - \mathrm{i}Q_3') \qquad & Q_3 &= \frac{-\mathrm{i}}{\sqrt{2}}(Q_2' + \mathrm{i}Q_3')
\end{aligned}$$

allows us to obtain

$$\frac{1}{2\mu}\sum_{j=1}^{4} P'^2_j - \frac{1}{2}\mu\omega^2\sum_{j=1}^{4} Q'^2_j = \pm \mathrm{i}\omega\sqrt{-\frac{2\mu\kappa^2}{E}}. \tag{32}$$

Denote $\omega' = \pm\mathrm{i}\omega$, (32) can be rewritten in the form

$$\frac{1}{2\mu}\sum_{j=1}^{4} P'^2_j + \frac{1}{2}\mu\omega'^2\sum_{j=1}^{4} Q'^2_j = \omega'\sqrt{-\frac{2\mu\kappa^2}{E}}. \tag{33}$$

The Schrödinger equation corresponding to (33) reads

$$\mathcal{H}_0\psi = \epsilon\psi \tag{34}$$

where we have denoted the left- and the right-hand sides of (33) by $\mathcal{H}_0$ and ϵ, respectively, which are the pseudo-Hamiltonian of a four-dimensional harmonic oscillator and pseudo-energy eigenvalue, respectively. Obviously, the eigenvalue is given by

$$\epsilon = (n_1 + n_2 + n_3 + n_4 + 2)\hbar\omega' \tag{35}$$

with $n_j = 0, 1, \ldots,$ for $j = 1$ to 4, so that one easily obtains

$$E = -\frac{\mu\kappa^2}{2}\frac{1}{n'^2} \tag{36}$$

where

$$n' = \tfrac{1}{2}(n_1 + n_2 + n_3 + n_4 + 2) = 1, \tfrac{3}{2}, 2, \ldots. \tag{37}$$

Similarly, it can be shown that the constraint (28) requires [20]

$$n_1 + n_2 = n_3 + n_4 \pm 2q \tag{38}$$

which is a monopole-dependent constraint condition. If, without a monopole, i.e. $q = 0$, it will reduce to the usual one, i.e. $\mathbf{L} \cdot \mathbf{R} \Longrightarrow 0$, whose physical meaning is only that the angular momentum vector and the PRL vector are orthogonal to each other.

Since q can be either positive or negative, after selecting one solution $n_1 + n_2 = n_3 + n_4 + 2q$, from equations (36) and (37) we have

$$E_n = -\frac{\mu\kappa^2}{2}\frac{1}{(2n + 1 - q)^2} \qquad n = (n_1 + n_2)/2 = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots \tag{39}$$

or

$$E_m = -\frac{\mu\kappa^2}{2}\frac{1}{(2m + 1 + q)^2} \qquad m = (n_3 + n_4)/2 = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots \tag{40}$$

which coincide with (22) and (23). Consequently, the monopole-hydrogen atom system (11) can be connected to a four-dimensional harmonic oscillator (33) with a monopole-dependent constraint condition shown by (28).

4. Monopole harmonics and shift operator of $\mathbf{L}^2$

In this section, let us return to the angular part of wavefunctions of H given in (11). In this case the usual spherical harmonics $Y_{l,m}(\theta, \phi)$ should be replaced by the monopole harmonics $Y_{q,l,m}(\theta, \phi)$, which are simultaneous eigensections of $\mathbf{L}^2$ and L_z with eigenvalues $l(l + 1)$ and m,

$$\mathbf{L}^2 Y_{q,l,m} = l(l + 1)Y_{q,l,m} \qquad L_z Y_{q,l,m} = mY_{q,l,m} \tag{41}$$

where $l = |q|, |q| + 1, |q| + 2, \ldots$, and for each value l, m ranges from $-l$ to $+l$ in integer steps of increment [24]. Explicit expressions for $Y_{q,l,m} = \Theta_{q,l,m}(\theta)\Phi_{q,m}(\phi)$ were given in [24] by dealing with some partial differential equations. For instance, in region R_a (see [24], the vector potential $\mathbf{A}$ is defined in two different regions R_a and R_b), $\Phi_{q,m}(\phi) = \mathrm{e}^{\mathrm{i}(m+q)\phi}$, which is obtained from

$$L_z \Phi_{q,m}(\phi) = (-\mathrm{i}\frac{\partial}{\partial\phi} - q)\Phi_{q,m}(\phi) = m\Phi_{q,m}(\phi) \tag{42}$$

at the same time, $\Theta_{q,l,m}(\theta)$ is obtained through solving

$$[l(l + 1) - q^2]\Theta_{q,l,m}(\theta) = \left[-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}(m + q\cos\theta)^2\right]\Theta_{q,l,m}(\theta). \tag{43}$$

Now, we want to rederive monopole harmonics by the operator method, without dealing with the second-order partial differential equations as shown by (43). What we have to do is to find monopole-dependent shift operators of $\mathbf{L}^2$ and L_z. As usual, the angular momentum operator satisfies

$$[L_z, L_\pm] = \pm L_\pm \qquad [L_+, L_-] = 2L_z \tag{44}$$

thus, $L_\pm = L_x \pm \mathrm{i}L_y$ are shift operators of L_z, they shift the quantum number m as follows:

$$L_\pm Y_{q,l,m} = \sqrt{[l(l + 1) - m(m \pm 1)]}Y_{q,l,m\pm1}. \tag{45}$$

Shift operators for $\mathbf{L}^2$ have been discussed in [23] that can change the quantum number l. With the shift operators the usual spherical harmonics are obtained. Following in the same spirit, we construct shift operators of $\mathbf{L}^2$ in the presence of a monopole.

In the derivation of these shift operators, the scalar and vector properties of operators play an important role. By definition, K is a scalar operator with respect to $\mathbf{L}$ if

$$[L_\alpha, K] = 0 \qquad \alpha = x, y, z \tag{46}$$

and $\mathbf{V}$ is a vector operator with respect to $\mathbf{L}$ if

$$[L_\alpha, V_\beta] = \mathrm{i}\epsilon_{\alpha\beta\gamma} V_\gamma. \tag{47}$$

From (47) it follows that

$$[\mathbf{L}^2, \mathbf{V}] = 2\mathbf{V} + 2\mathrm{i}\mathbf{V} \times \mathbf{L}. \tag{48}$$

It can be verified that

$$[L_\alpha, (\mathbf{V} \times \mathbf{L})_\beta] = \mathrm{i}\epsilon_{\alpha\beta\gamma} (\mathbf{V} \times \mathbf{L})_\gamma \tag{49}$$

which means that $\mathbf{V} \times \mathbf{L}$ is also a vector operator to $\mathbf{L}$. Replacing $\mathbf{V}$ by $\mathbf{V} \times \mathbf{L}$ in (48) one gets

$$[\mathbf{L}^2, \mathbf{V} \times \mathbf{L}] = 2\mathbf{V} \times \mathbf{L} + 2\mathrm{i}(\mathbf{V} \times \mathbf{L}) \times \mathbf{L}. \tag{50}$$

With

$$(\mathbf{V} \times \mathbf{L}) \times \mathbf{L} = -\mathbf{V}\mathbf{L}^2 + \mathrm{i}(\mathbf{V} \times \mathbf{L}) + (\mathbf{V} \cdot \mathbf{L})\mathbf{L} \tag{51}$$

we then have

$$[\mathbf{L}^2, \mathbf{V} \times \mathbf{L}] = 2\mathrm{i}(-\mathbf{V}\mathbf{L}^2 + (\mathbf{V} \cdot \mathbf{L})\mathbf{L}). \tag{52}$$

Since $\mathbf{V} \cdot \mathbf{L}$ is a scalar operator with respect to $\mathbf{L}$, so that

$$[\mathbf{L}^2, \mathbf{V} \cdot \mathbf{L}] = 0. \tag{53}$$

Now consider the operators

$$\mathbf{U}^\pm = \pm\mathrm{i}(\mathbf{V} \times \mathbf{L}) + \mathbf{V}K^\pm - (\mathbf{V} \cdot \mathbf{L})\mathbf{L}\frac{1}{K^\pm} \tag{54}$$

with

$$K^\pm = \sqrt{\mathbf{L}^2 + \tfrac{1}{4}} \pm \tfrac{1}{2} \tag{55}$$

which are scalar operators. (We have placed $K^\pm$ to the right of $\mathbf{V}$ and $\mathbf{L}$ to allowed them to operate directly on eigenfunction $Y_{q,l,m}$. This simplifies the calculations.) Hence $\mathbf{U}^\pm$ are vector operators and

$$[L_\alpha, (\mathbf{U}^\pm)_\beta] = \mathrm{i}\epsilon_{\alpha\beta\gamma} (\mathbf{U}^\pm)_\gamma. \tag{56}$$

Due to equations (48),(50) and (53), one obtains

$$[\mathbf{L}^2, \mathbf{U}^\pm] = \pm 2\mathbf{U}^\pm K^\pm. \tag{57}$$

Let $D_\pm$ denote the z-component of $\mathbf{U}^\pm$, it is easy to get

$$[\mathbf{L}^2, D_\pm] = \pm 2 D_\pm K^\pm \qquad [L_z, D_\pm] = 0. \tag{58}$$

Obviously, $D_\pm$ are the wanted shift operators that only change the quantum number l as follows

$$D_\pm Y_{q,l,m} \longrightarrow Y_{q,l\pm 1,m}. \tag{59}$$

Consequently, the operators $D_\pm$ together with $L_\pm$ can generate all arbitrary monopole harmonics from a starting one, e.g.

$$Y_{q,l,m} = C_{lm} L_+^m D_+^l Y_{q,0,0} \tag{60}$$

where C_{lm} are the normalized constants.

We now give explicit expressions for the shift operators $D_\pm$ discussed above. Let $\mathbf{V} = r^{-1}\mathbf{r}$, then

$$\mathbf{V}\cdot\mathbf{L} = \mathbf{L}\cdot\mathbf{V} = -q. \tag{61}$$

With

$$V_z = \cos\theta \qquad \mathrm{i}(\mathbf{V}\times\mathbf{L})_z = \sin\theta\frac{\partial}{\partial\theta} \tag{62}$$

one finally obtains

$$D_\pm = \pm\sin\theta\frac{\partial}{\partial\theta} + \cos\theta K^\pm + qL_z\frac{1}{K^\pm} \tag{63}$$

where $L_z = -\mathrm{i}\frac{\partial}{\partial\phi} - q$ acts in region R_a, whereas $L_z = -\mathrm{i}\frac{\partial}{\partial\phi} + q$ in region R_b. We emphasize that $D_\pm$ are the monopole-dependent extension of the shift operators given in [23].

As special cases of equation (59), D_- annihilate $Y_{q,l,\pm l}$, we have

$$D_-Y_{q,l,l} = D_-\Theta_{q,l,l}(\theta)\Phi_{q,m=l}(\phi) = 0. \tag{64}$$

Due to

$$K^\pm Y_{q,l,m} = (l + \tfrac{1}{2} \pm \tfrac{1}{2})Y_{q,l,m} \qquad L_zY_{q,l,l} = lY_{q,l,l} \tag{65}$$

from equations (63) and (64) we have

$$D_-\Theta_{q,l,l}(\theta) = \left(-\sin\theta\frac{\partial}{\partial\theta} + l\cos\theta + q\right)\Theta_{q,l,l}(\theta) = 0 \tag{66}$$

hence,

$$\Theta_{q,l,l}(\theta) = C_{ll}(\sin\theta)^l\left(\frac{1-\cos\theta}{1+\cos\theta}\right)^{q/2}. \tag{67}$$

The results coincide with those obtained from (43). By another manner, an arbitrary monopole harmonics can also be obtained through

$$Y_{q,l,m} = C_{lm}L_-^{l-m}Y_{q,l,l}. \tag{68}$$

Consequently, monopole harmonics are obtained from the operator method. When $q = 0$, they reduce naturally to the usual spherical harmonics.

Eventually, we would like to briefly state the radial part of the Hamiltonian H, since (see [24])

$$\vec{\pi}^2 = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}[\mathbf{L}^2 - q^2] \tag{69}$$

and we take the coordinate-space wavefunctions

$$\psi_{q,\lambda,l,m}(\mathbf{r}) = R_{q,\lambda,l}(r)Y_{q,l,m}(\theta,\phi) \tag{70}$$

from equation (11) it follows

$$\left[-\frac{1}{2\mu r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{l(l+1)}{2\mu r^2} - \frac{\kappa}{r}\right]R_{q,\lambda,l}(r) = E_\lambda R_{q,\lambda,l}(r). \tag{71}$$

Evidently, the form of the radial wave equation (71) is the same as that of the usual hydrogen atom, except for the different values of $l = |q|, |q|+1, |q|+2, \ldots$. The solution of (71) is

$$R_{q,\lambda,l}(r) = C_{\lambda,l}r^l\mathrm{e}^{-\sqrt{-2\mu E_\lambda}r}\,{}_1F_1\left(l+1-\lambda; 2l+2; 2\sqrt{-2\mu E_\lambda}r\right) \tag{72}$$

where $C_{\lambda,l}$ are normalized constants, ${}_1F_1(\alpha;\beta;z)$ is a confluent hypergeometric function, and

$$E_\lambda = -\frac{\mu\kappa^2}{2}\frac{1}{\lambda^2}. \tag{73}$$

Further analyses show that the principle quantum number

$$\lambda = |q|+1, |q|+2, |q|+3, \ldots \tag{74}$$

and

$$l = |q|, |q|+1, \ldots, \lambda - 1. \tag{75}$$

The smallest value of λ is $|q|+1$, then we must be aware that in the energy spectrum given by (22) or (39), the value of n takes

$$n = (|q|+q)/2, (|q|+q+1)/2, (|q|+q+2)/2, \ldots. \tag{76}$$

In summary, we have established the Hamiltonian of a hydrogen atom with a monopole, conserved quantities **L** and **R** generate an $SO(4)$ dynamical symmetry of the system. Moreover, the system can be connected to a four-dimensional harmonic oscillator with a monopole-dependent constraint condition. In addition, due to the shift operator of $\mathbf{L}^2$, the monopole harmonics can be obtained completely by the operator method.

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